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## DESIGNING OF NON-CIRCULAR GEARS

In this paper, the authors present methods for designing of non-circular gears, including internal and external gears with spur or helical teeth. Technology related issues that determine tooth profile calculation algorithm are described. The results presented in this paper can become groundwork for further investigations of other particular properties of non-circular gears, similar to investigations of spur, helical and bevel gears. Examples of such properties include kinematics and application of special purpose gears or issues related to strength, dynamics, tribology, etc.

### 1. Introduction

The idea of non-circular gears originates from the precursors of the engineering thought. These gears were sketched by Leonardo da Vinci, and found their application in many types of mechanical devices, like clocks and toys. In late XIX, century Franz Reuleaux ordered at Gustav Voigt Mechanische Werkstatt in Berlin a series of non-circular gear models to help study kinematics. The gears made at those times had simplified tooth shapes and, for this reason, the meshing conditions were not always correct (Fig. 1).

The applications of non-circular gears include:

- Textile industry machines, for improving machine kinematics resulting in the process optimization [3], [6]
- Window shade panel drives, for introducing vibration which interfere with natural oscillations and cancel them out [9]
- Mechanical presses, for optimization of work cycle kinematics [1], [6]
- High torque hydraulic engines for bulkhead drives [12]
- High-power starters, mechanical systems providing progressive torque for easier starting of the machines, where progressive torque helps to overcome the start-up inertia

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- Forging machines, for optimizing the work cycle parameters (reducing pressure dwell time) [2]



Fig. 1. A historical model of a non-circular gear

## 2. Geometry of meshing

The industry standard [7][7] involute tooth shape has been chosen for the use in non-circular gears and therefore the existing involute gearing standards and methods can be adopted and used.

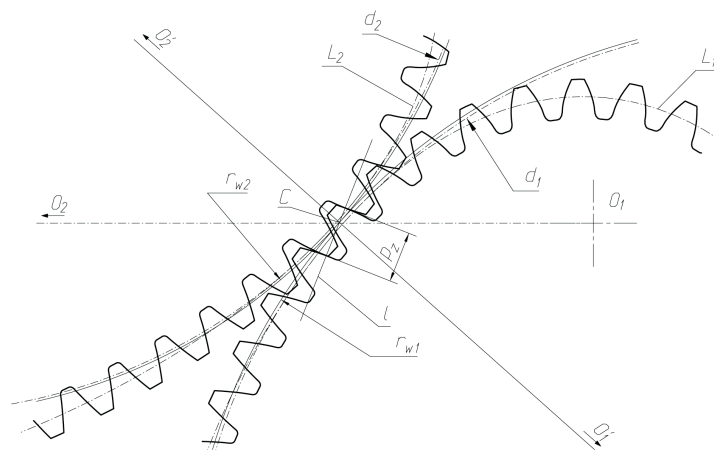


Fig. 2. Non-circular gear's meshing geometry

Fig. 2 shows non-circular segments meshed. The axle distance is increased, and a backlash is created. The roll lines  $L_1$  and  $L_2$ , correspond to

pitch circles in regular gears. The temporary line of action  $l$  meets teeth contact points at right angles to the flanks. The line  $O'_1O'_2$ , perpendicular to the lines  $L_1$  and  $L_2$  where their distance is minimal, passes through the centers  $O'_1O'_2$  of replacement pitch circles  $d_1$  and  $d_2$  and roll circles with radii  $r_{w1}$  and  $r_{w2}$ . The roll circles touch at a temporary center of meshing  $C$ , where the lines  $O'_1O'_2$  and  $l$  intersect.

In the case of non-circular gear, each flank of each tooth can have different shape. If the gear (consists of segments of a circular gear or a rack) some of the flanks may have similar shapes, but this is a rare, special case. A free-form cutting technology is often used instead of hobbing or shaping. The exact profile of the teeth must then be specified, as it is not defined by any specific curve, and such a profile is not easy to find. Therefore, it is usually required to include the exact profile in the documentation (preferably as a numerical model), especially when the part will be checked using the coordinate measuring technology.

### 3. Roll lines

The roll lines correspond to pitch circles in regular gears. They represent a non-circular gear as two rollers rolling together without slip, provided there is no addendum modification and the nominal axle distance is used. Roll lines are divided in  $z$  parts that are  $p$  long, where  $z$  is the gear's number of teeth, and  $p$  is the pitch.

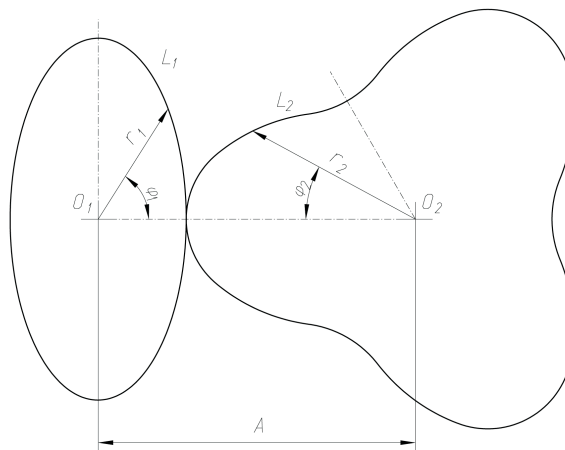


Fig. 3. Non-circular roll lines example

#### 4. Basic equations

The gear is represented by two roll lines  $L_1$  and  $L_2$  with centers in  $O_1$  and  $O_2$ , rolling together without slip. (Fig. 3).

The  $L_1$  line is described in polar coordinates as  $r_1(\varphi_1)$ . The gear's ratio is described as:

$$v(\varphi_1) = \frac{\omega_1(\varphi_1)}{\omega_2(\varphi_1)} = \frac{r_2(\varphi_1)}{r_1(\varphi_1)} \quad (1)$$

where  $\omega_1(\varphi_1)$  and  $\omega_2(\varphi_1)$  are angular velocity functions for gear 1 and 2 accordingly.

The distance  $A$  between the centers of roll lines is constant:

$$A = r_1(\varphi_1) + r_2(\varphi_1) = \text{const} \quad (2)$$

The roll lines roll without slip, therefore:

$$d\varphi_2 = \frac{r_1(\varphi_1)}{r_2(\varphi_1)} d\varphi_1 \quad (3)$$

Taking (1), (2) and (3) into consideration, after integrating we have:

$$\varphi_2 = \int_0^{\varphi_1} \frac{r_1(\varphi_1)}{A - r_1(\varphi_1)} d\varphi_1 \quad (4)$$

The integration constant is found from the condition that if  $\varphi_1 = 0$  then  $\varphi_2 = 0$ .

In the example shown in Fig. 3, the period  $T$  of the ratio function  $v(\varphi_1)$  equals  $\pi$ ,  $T_1=2\pi$ , and  $T_2=3\pi$ , so when  $\varphi_1 = \pi$  then  $\varphi_2 = 2/3\pi$ . Therefore:

$$\frac{2}{3}\pi = \int_0^{\pi} \frac{r_1(\varphi_1)}{A - r_1(\varphi_1)} d\varphi_1 \quad (5)$$

The axle distance  $A$  can be found from equation (4). Line  $L_2 = r_2(\varphi_1)$  is found from (2) and (3). In order to transmit the torque continuously, the gear's ratio function must be periodic, and its period  $T$  must correspond to the periods  $T_1$  and  $T_2$  of gears 1 and 2 as follows:

$$T = \frac{T_1}{n_2} = \frac{T_2}{n_1} \quad (6)$$

where  $n_1$  and  $n_2$  are natural numbers.

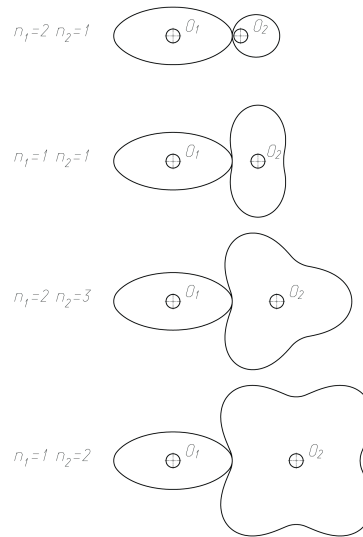
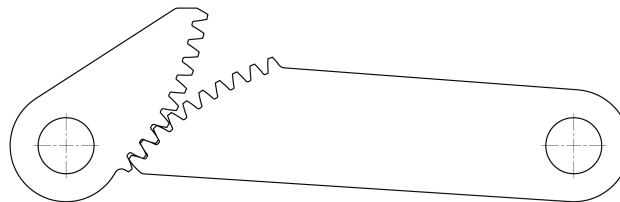
Fig. 4. Roll lines with different  $n$  parameters

Fig. 5. Aperiodic non-circular gear

In the case of an aperiodic non-circular gear, angular positions of the members are limited. Usually, a specific ratio function is used, for example a logarithmic function.

Standard curves can be used as roll lines as long as they fulfill the requirements described above. The most frequently used one is the ellipse. In order to represent a free-form roll line, it is first divided into many little segments. Each segment is represented as a segment of an Archimedean spiral. In this spiral, the distance from the center is a linear function of the spiral angle, which makes this curve naturally suitable for describing the non-circular gears, although other curves can be used as well.

The equation of Archimedean spiral is:

$$r = k\alpha, \quad (7)$$

where  $k$  is the spiral's parameter.

Let us assume that the  $L_1$  roll line is given, or the ratio function is given and we can find  $L_1$  from (1), and we look for the  $L_2$  line.  $L_1$  is divided into  $i$  little segments  $l_{1i}$  so that each of those segments can be represented by an Archimedean spiral with adequate precision.

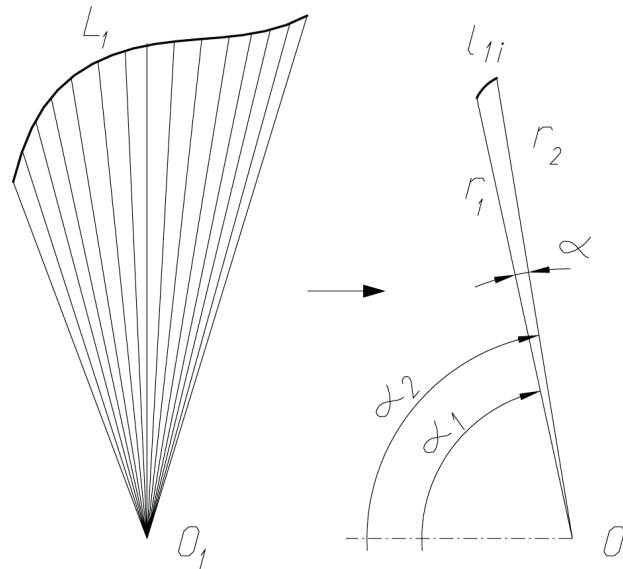


Fig. 6. Roll line divided into segments

$r_1$ ,  $r_2$ ,  $\alpha$ ,  $\alpha_1$  and  $\alpha_2$  are parameters of a current segment  $l_{1i}$  and mean, respectively:

$r_1$  – spiral's starting radius

$r_2$  – spiral's ending radius

$\alpha$  – spiral segment's whole angle

$\alpha_1$  – spiral's starting angle

$\alpha_2$  – spiral's ending angle

These parameters will be used only for the currently processed segment, so there is no need to index them.

As shown in Fig. 6:

$$\alpha = \alpha_2 - \alpha_1 \quad (8)$$

Taking (9) into account we have:

$$\alpha_1 = \frac{r_1}{k} \alpha_2 = \frac{r_2}{k}$$

$$k = \frac{r_2 - r_1}{\alpha}$$

$$dl = \sqrt{(dr)^2 + (r d\alpha)^2} = d(k\alpha) = k d\alpha$$

$$dl = \sqrt{(k d\alpha)^2 + (r d\alpha)^2} = \sqrt{(k d\alpha)^2 + (k \alpha d\alpha)^2} = k \sqrt{1 + \alpha^2} d\alpha$$

$$\int dl = \frac{k}{2} [\alpha \sqrt{\alpha^2 + 1} + \ln(\alpha + \sqrt{\alpha^2 + 1})]$$

$$l_{1i} = \int_{\alpha_1}^{\alpha_2} dl = \frac{k}{2} [\alpha_2 \sqrt{\alpha_2^2 + 1} + \ln(\alpha_2 + \sqrt{\alpha_2^2 + 1}) - \alpha_1 \sqrt{\alpha_1^2 + 1} - \ln(\alpha_1 + \sqrt{\alpha_1^2 + 1})] \quad (9)$$

The lines  $L_1$  and  $L_2$  roll without slip, and the line  $L_2$  is divided, similarly as  $L_1$ , into  $il_{2i}$ , segments. Therefore, from (3) it follows:

$$|l_{1i}| = |l_{2i}| \quad (10)$$

Using equations (2)-(12), we can find the segments  $l_{2i}$ , and consequently the  $L_2$  line.

The presented procedure usually requires a lot of calculations and storing of arrays of co-efficients and other data. For this reasons it is usually implemented as a computer application.

In some cases, the properties of Archimedean spiral make it possible to divide the roll line only into a couple of segments with all the constructional requirements fulfilled.

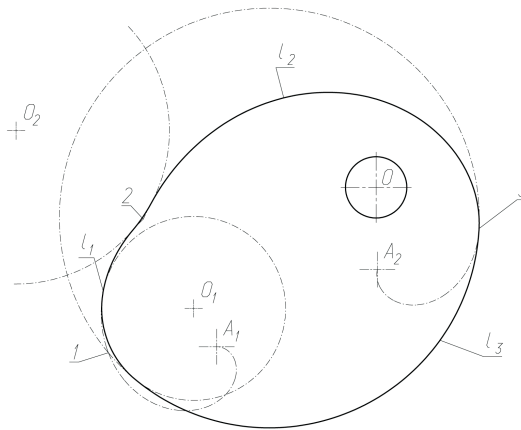


Fig. 7. A roll line divided into segments

Fig. 7 shows a roll line, which can be constructed by using three elements—two segments  $l_1$  and  $l_2$  of Archimedean spirals  $A_1$  and  $A_2$  and a circular segment  $l_3$ . One can notice that at points 1 and 2 the touching segments are not tangent, and the resulting roll line has no derivative in these points. Therefore, a step change of the gear ratio occurs there, and the transfer of power is not uniform. The inertia of the driven mass leads to creation of excessive dynamic forces. In order to avoid that effect, smooth transitions between the lines must be secured by introducing tangent segments of the curves. In the example above, it is accomplished by introducing tangent circles  $O_1$  and  $O_2$ . There are no special restrictions on types of the curves used for this purpose. The most frequently used ones are conic sections (ellipse, parabola, hyperbola), polynomial curves, exponential, power and logarithmic curves as well as Bezier curves, B-spline curves and NURBS.

### 5. Finding tooth shapes

In order to avoid problems with meshing of the gears, the teeth profiles should properly mesh with a reference rack. This condition is satisfied when the reference rack's pitch line is rolling over the gear's roll line.

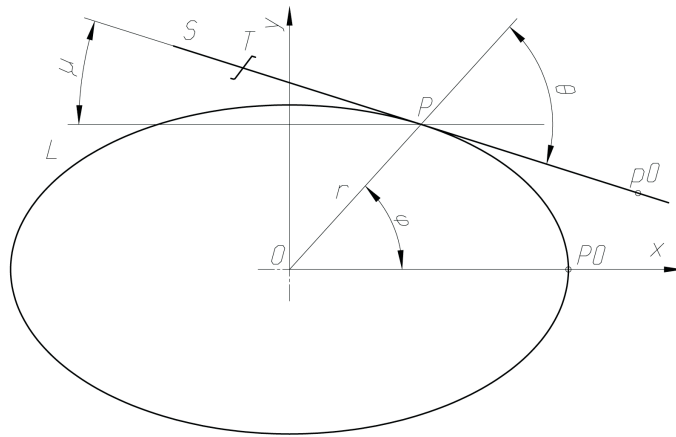


Fig. 8. Reference rack's pitch line of rack rolling over a non-circular roll line

The non-circular roll line  $L$  is described as  $L = r(\varphi)$ . The reference rack pitch line  $S$  with profile  $T$  rolls over  $L$ . The coordinates of contact point  $P$  are:

$$P[x, y] = [r(\varphi)\cos(\varphi), r(\varphi)\sin(\varphi)] \quad (11)$$



The angle between  $r(\varphi)$  and the reference rack's pitch line  $S$  equals:

$$\theta = \arctan\left(\frac{r(\varphi)}{\frac{dr}{d\varphi}}\right). \quad (12)$$

Therefore:

$$\mu = \varphi - \theta \quad (13)$$

If  $\varphi=0$   $P0=p0$ ,  $S$  rolls over  $L$  without slip from  $P0$  to  $P$ , therefore the length of segment  $l$  of the line  $L$  between  $P0$  and  $P$  is equal to the length of the segment  $s$  between  $p0$  and  $P$  of the line  $S$ .

If  $L$  consists of  $i$  segments  $l_i$ , the analyzed point  $P$  corresponds to the segments' lengths  $l_j$ ,  $l_i$ , and the parameters  $k_i$  are found to from (11), then:

$$|s| = \sum_{n=1}^j l_n \quad (14)$$

The roll line  $L$  is divided into  $z$  segments, the length of each of them is  $p$ . The module  $m$  is found from the equation:

$$m = \frac{|L|}{\pi z} = \frac{\sum_{n=1}^i l_n}{\pi z}, \quad (15)$$

where  $z$  is the number of teeth in the gear.

## 6. Context analysis method

The context analysis method consists in running the system through a series of states and recording them as the context, from which the data fulfilling certain criteria are selected, and these constitute the result of the analysis. The context is created as the positions of points on the reference profile rolling with the reference pitch line of the rack's over the gear's roll line. Analytic geometry is used to find positions of these points.

The reference profiler of the rack's meshing with the gear can never collide into itthe latter one. Therefore, any geometric locus of the system where the reference profile is located in any of the intermediate states must not belong to the gear. The border of the area constituted by these loci is the gear's flank (see Fig. 9).

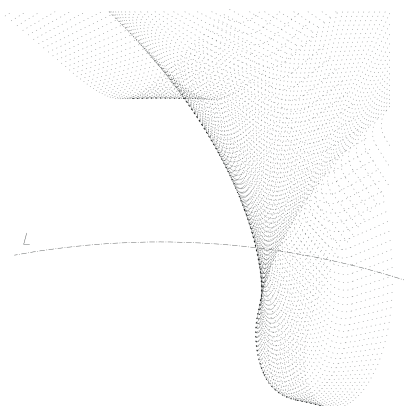


Fig. 9. Gear's flank of gear found using context analysis method

In the case of an internal gear, the material is found on the opposite side of the profile. Usually, for proper meshing, some additional requirements need to be fulfilled.

The method described above allows for very easy modifications of the profile. Such modifications, similarly as in the case of regular gears, are in most cases required for meshing optimization. Typical examples include profile shift for avoiding the interference in an internal gear, profile and line crowning for improvement of the meshing pattern, or profile shift for improving the slide/roll ratio. The range of a modification can be wide and applied individually to each tooth, as in the case of non-circular gears a tooth will mesh with determined, corresponding tooth space.

## 7. Replacement gear

A non-circular gear remaining in a defined, momentary state can be represented as a circular gear, whose meshing parameters represent temporary meshing parameters of a non-circular gear. A change in the position of non-circular gear's will, in most cases, change the temporary meshing parameters, so that a number of states must be analyzed to create the meshing parameter characteristic of the gear.

Fig. 10 shows a non-circular gear in a defined angular position described by  $\alpha_1$  and  $\alpha_2$ , gear rotation angles and the replacement circular gear consisting of two gears with centers  $O'_1$  and  $O'_2$ , and with pitch radii  $\rho_1$  and  $\rho_2$ . These radii are equal to curvatures of roll lines in the contact point and can be calculated from the equation:

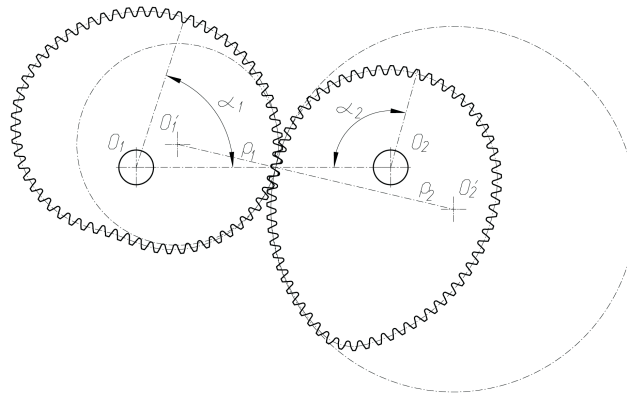


Fig. 10. Non-circular gear and the corresponding circular replacement gear

$$\rho = \frac{\left[ r^2 + \left( \frac{dr}{d\varphi} \right)^2 \right]^{\frac{3}{2}}}{r^2 + 2 \left( \frac{dr}{d\varphi} \right)^2 - r \frac{d^2r}{d\varphi^2}},$$

assuming the roll line radii are described in polar coordinates by functions  $r(\varphi)$ .

When a digital computing method is used, it is more convenient to calculate the curvature by finding a circle over 3 points  $P$  on the roll line close to the contact point.:

$$P_1(a, b),$$

$$P_2(c, d),$$

$$P_3(e, f).$$

We look for the radius  $r$  of the circle over points  $P_1$ ,  $P_2$  and  $P_3$  with the center in point  $O(h, k)$ . In the right triangle we have:

$$(a - h)^2 + (b - k)^2 = r^2,$$

$$(c - h)^2 + (d - k)^2 = r^2,$$

$$(e - h)^2 + (f - k)^2 = r^2.$$

After transformations:

$$k = \frac{1}{2} \frac{(a^2 + b^2)(e - c) + (c^2 + d^2)(a - e) + (e^2 + f^2)(c - a)}{b(e - c) + d(a - e) + f(c - a)},$$

$$h = \frac{1}{2} \frac{(a^2 + b^2)(f - d) + (c^2 + d^2)(b - f) + (e^2 + f^2)(d - b)}{a(f - d) + c(b - f) + e(d - b)},$$

$$r = \sqrt{(a - h)^2 + (b - k)^2}.$$

For analyzing the meshing parameters, it is convenient to calculate the pitch radii  $r_1$  and  $r_2$  of the replacement gears for many positions, and to create a function showing how those radii change in the function of the rotation angle  $\alpha_1$  of the non-circular gear 1. An example of such a function is shown below.:

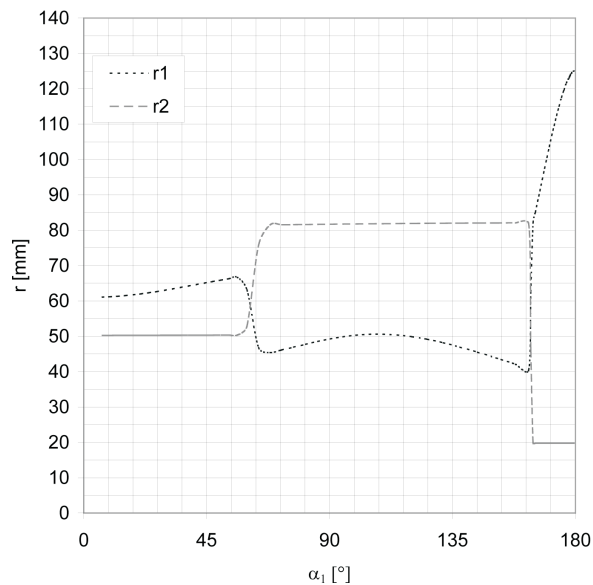


Fig. 11. Example of function of replacement gear pitch radii

## 8. Non-circular helical gears

Teeth lines of helical gears cross the generating lines of the pitch cylinder at helix angle  $\beta$ . Two meshed helical gears, that have parallel axes, have the same  $\beta$  angles, but their directions are different – one of them is left, and the other one is right.

In the case of helical gears, the generating rack is skew to the generating lines of a cylindrical surface. The teeth flanks are enveloped by an additionally rotated and shifted generating rack.

Meshing of helical gears is analyzed in transverse plane [7]. If the reference rack  $s_n$  is defined in normal plane (standard in Europe), a conversion of the reference rack  $s_t$  into a transverse plane must be carried out:

$$p_n = p_t \cos \beta,$$

$$m_n = m_t \cos \beta,$$

$$tg \alpha_t = \frac{tg \alpha_n}{\cos \beta},$$

where  $p$  is pitch,  $m$  is module,  $\alpha$  is profile angle, and  $\beta$  is helical angle.

Helical gear's geometry is created by means of the methods described above. An example isometric drawing of a non-circular helical gear is shown below.

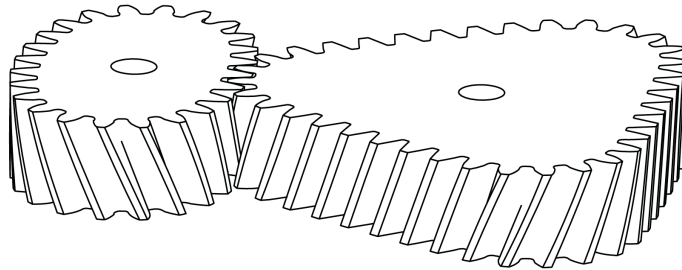


Fig. 12. Non-circular helical gear

## 9. Verification of non-circular gear designing method

In order to verify the presented procedure, sample gears were manufactured and examined.



Fig. 13. Non-circular gear on checking fixture

The first manufactured gear was a simple elliptical design. The gear was checked for smooth rotation, and its ratio function was compared to the theoretical one. No design-related problems were found.

The second gear was a one with changing profile shift, having different shift on every tooth (it was possible because in this design a tooth will mesh with only one corresponding tooth space).

Table 1.

Gear parameters- profile shift changes

Parameter	Symbol	Value
Module	m	4,71
Nr of teeth	z	32
Pressure angle	$\alpha$	20°
Profile shift coefficient	x	-0,2 – 0,2
Tooth height coefficient	y	1
Tip clearance coefficient	c	0,25
Tooth thickness on reference diameter	g	6,710 – 8,081

Table 2.

Profile shifts on respective teeth

Tooth nr	x	Tooth nr	x	Tooth nr	x	Tooth nr	x
1	0	9	0	17	0	25	0
2	-0,1	10	0,1	18	-0,1	26	0,1
3	-0,2	11	0,2	19	-0,2	27	0,2
4	-0,1	12	0,1	20	-0,1	28	0,1
5	0	13	0	21	0	29	0
6	0,1	14	-0,1	22	0,1	30	-0,1
7	0,2	15	-0,2	23	0,2	31	-0,2
8	0,1	16	-0,1	24	0,1	32	-0,1

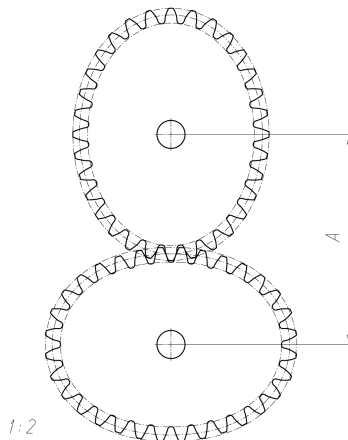


Fig. 14. Gear profile- profile shift changes

The third gear had different pressure angles on left and right flanks. The left and the right flanks were meshing as the gear rotat counter clockwise, - and clockwise, respectively.

Table 3.

Gear parameters- changing pressure angle

Parameter	Symbol	Value
Module	m	4,71
Nr of teeth	z	32
Pressure angle	$\alpha$	17,5°/22,5°
Profile shift	x	0
Tooth height coefficient	y	1
Tip clearance coefficient	c	0,25
Tooth thickness on reference diameter	g	7,396

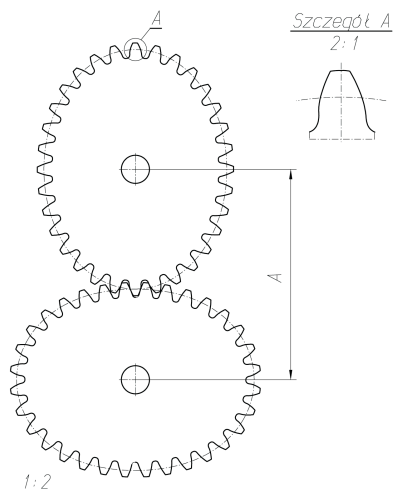


Fig. 15. Gear profile- changing pressure angle

The fourth gear was with concave segments. One of the gears had the shape of a rounded square. If a correct axle distance and rounding radius ratio is used, the meshing gear has concave segments. The axle distance was 150 mm, and the rounding radius was 25 mm.



Table 4.

Gear parameters- concave segments

Parameter	Symbol	Value
Module	m	4,91
Nr of teeth	z	32
Pressure angle	$\alpha$	20°
Profile shift	x	0
Tooth height coefficient	y	1
Tip clearance coefficient	c	0,25
Tooth thickness on reference diameter	g	7,714

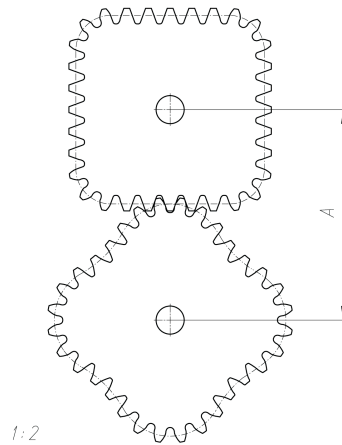


Fig. 16. Gear profile- concave segments

## 10. Conclusions

- The described method of design of non-circular gear allows for free shaping of the gear and for optimizing its meshing parameters
- The presented practical verification shows that the use of the method can yield good results. The manufactured gears had the expected ratio function and no problems with gear's meshing were observed. In order to increase the precision in checking the ratio function, one should use more accurate gears (eg. WEDM or precision CNC cut gears). Professional encoders and an adequate equipment for experiment data processing should be used.
- The results presented in this paper can constitute fundamentals for further investigations of other particular properties of non-circular gears, such as

the investigations of spur, helical and bevel gears. These investigations may include, for example, gear kinematics and application of special purpose gears, or may concern issues related with strength, dynamics, tribology etc..

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#### **Projektowanie walcowych przekładni zębatych o zmieniającym się przełożeniu**

#### Streszczenie

W artykule przedstawiono metodykę projektowania przekładni walcowych o zmieniającym się przełożeniu z uwzględnieniem uzębień zewnętrznych i wewnętrznych o prostej lub skośnej linii zęba. Uwzględniono aspekty technologiczne, które determinują algorytm obliczeń zarysów zębów. Wyniki niniejszego opracowania mogą stanowić podstawę do dalszego badania różnych konkretnych własności przekładni zębatych o zmieniającym się przełożeniu, analogicznie do prowadzonych badań przekładni zębatych walcowych i stożkowych. Przykładami mogą tu być zagadnienia związane z kinematyką i zastosowaniem przekładni specjalnego przeznaczenia, jak również zagadnienia wytrzymałościowe, dynamiczne, trybologiczne i inne.